TWO TAUBERIAN THEOREMS FOR THE PRODUCT OF ABEL AND CESÀRO SUMMABILITY METHODS

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Abstract. In this work, we prove two Tauberian theorems, which develop the classical type Tauberian theorems for the method of Abel summability, to get Cesàro summability of a sequence out of the product methods of Abel and Cesàro summability of the sequence.

1. Introduction

$(A)(C,\alpha)$ summability method was introduced independently from each other with different notations by Kogbetliantz [1], Lord [2] and Jakimovski [3]. Then they obtained some theorems associated with this method. Later than a number of authors such as Erdem [4], Erdem and Çanak [5], Erdem and Çanak [6], Erdem and Totur [7], Çanak and Erdem [8], Erdem and Çanak [9], Çanak et al. [10], Pati [11] give some Tauberian theorems about this method in summability theory.

For a real sequence $u=(u_n)$, the (C,α) means of the sequence (u_n) are defined by

$$\sigma_n^\alpha(u)=\frac{S_n^\alpha}{A_n^\alpha}=\frac{1}{A_n^\alpha}\sum_{j=0}^{n}A_{n-j}^{\alpha-1}u_j.$$

A sequence (u_n) is said to be (C,α) summable to a finite number ξ, where $\alpha>-1$ if

$$\lim_n \sigma_n^\alpha(u)=\xi, \tag{1.1}$$

and we write $u_n\to\xi\ (C,\alpha)$.

Kogbetliantz [12] proved the identity

$$\tau_n^\alpha(u)=n\Delta\sigma_n^\alpha(u). \tag{1.2}$$

Kogbetliantz [1] also proved the identity

$$(\alpha+1)\left(\sigma_n^\alpha(u)-\sigma_n^{\alpha+1}(u)\right)=\tau_n^{\alpha+1}(u) \tag{1.3}$$

and it is used in the various steps of our proofs.

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Let A_n^α be defined by the generating function $(1-x)^{-\alpha-1} = \sum_{n=0}^{\infty} A_n^\alpha x^n$, $(|x|<1)$, where $\alpha > -1$. For a real sequence $u=(u_n)$, the (C,α) means of the sequence (u_n) are defined by

$$\sigma_n^\alpha(u) = \frac{S_n^\alpha}{A_n^\alpha} = \frac{1}{A_n^\alpha}\sum_{j=0}^{n} A_{n-j}^{\alpha-1} u_j.$$

A sequence (u_n) is said to be (C,α) summable to a finite number ξ, where $\alpha > -1$ if

$$\lim_n \sigma_n^\alpha(u) = \xi, \tag{1.4}$$

and we write $u_n \to \xi\ (C,\alpha)$.

If we put $\alpha = 1$ in the definition of (C,α) mean of (u_n), we see that the sequence $\left(\sigma_n^1(u)\right) = (\sigma_n(u))$ is the sequence of arithmetic means of (u_n).

If we take $\alpha = 0$ in the identity (1.3), then we have

$$u_n - \sigma_n(u) = \tau_n^1(u),$$

where $\tau_n^1(u) = \frac{1}{n+1}\sum_{j=1}^{n} j\Delta u_j$. From this, we easily see that $n\Delta\sigma_n(u) = \tau_n^1(u)$.

During this study, the symbol $[\lambda n]$ denotes the integral part of λn.

2. A brief history of Tauberian theory

In advance of progress to the proofs of major theorems, we require the next lemma to be used in the proofs of major theorems.

Lemma 1. *[13] If a sequence (u_n) is $(A)(C,\alpha+k)$ summable to ξ, then $\sigma_n^{(k)}(u^\alpha)$ is Abel (A) summable to ξ*

Lemma 2. *[13] If $p \geq 0, \alpha > -1, \beta > -1$ and a sequence $A_n (n \geq 0)$ satisfies the conditions*

$$\lim_{\delta\to 0^+} \liminf_{n\to\infty} \min_{n\leq m\leq (1+\delta)n} \frac{C_m^\alpha - C_n^\alpha}{n^p} \geq 0,$$

$$f_\beta(x) = o\{(1-x)^{-p}\} \text{ as } x \to 1^-$$

then

$$C_n^\alpha = o(n^p) \text{ as } n \to \infty.$$

Lemma 3. *[14] Assume that $u_n \to \xi(A)$. For $u_n \to \xi(C,1)$, it is necessary and sufficient condition that*

$$\lim_{\lambda\to 1^+} \liminf_{n\to\infty} \min_{n\geq m\geq \lambda n} \frac{1}{n}\sum_{n<k\geq m} u_k \leq 0.$$

3. Results

The purpose of this study is to innovate some latest Tauberian conditions for the $(A)(C,\alpha)$ summability method and so popularize a number of classical Tauberian theorems.

Theorem 1. *If (u_n) is (A)(C,β) summable to ξ and*

$$(n\Delta)_m \tau_n^{\alpha+m+1} = o(1) \tag{3.1}$$

then (u_n) is (C,α) summable to ξ.

Theorem 2. *If (u_n) is (A)(C,β) summable to ξ and*

$$\lim_{\lambda\to 1^+} \liminf_{n\to\infty} \min_{n<t\le[\lambda n]} \frac{1}{n} \sum_{n<k\le t} ((k\Delta)_m \tau_k^{\alpha+m}) \ge 0 \tag{3.2}$$

then (u_n) is (C,α) summable to ξ.

Proof of Theorem 1. Let θ be a positive number such that $\alpha+\theta\ge\beta$. By hypothesis, (u_n) is (A)(C, β) summable to ξ for any finite number ξ. Therefore, $u_n \to \xi$ $(A)(C,\alpha+\theta)$ and $\tau_n^{\alpha+\theta+1} \to 0$ (A) with similar operations

$$(n\Delta)_m \tau_n^{\alpha+\theta+m} \to 0 \ (A). \tag{3.3}$$

From (3.2),

$$(n\Delta)_m \tau_n^{\alpha+\theta+m} = o(1)$$

and

$$(n\Delta)(n\Delta)_{m-1} \tau_n^{\alpha+\theta+m} = o(1). \tag{3.4}$$

Then, by (3.3) and (3.4), we get

$$(n\Delta)_{m-1} \tau_n^{\alpha+\theta+m} = o(1) \tag{3.5}$$

by Tauber's first theorem [15]. Since (3.4) and (3.5) are satisfied, then we get

$$(n\Delta)_{m-1} \tau_n^{\alpha+\theta+m-1} = o(1)$$

from the identity

$$(\alpha+\theta+m)((n\Delta)_{m-1}\tau_n^{\alpha+\theta+m-1} - (n\Delta)_{m-1}\tau_n^{\alpha+\theta+m}) = (n\Delta)_m \tau_n^{\alpha+\theta+m}.$$

Continuing in this way, we have,

$$n\Delta\tau_n^{\alpha+\theta+1} = o(1).$$

It follows from this equation and $\tau_n^{\alpha+\theta+1} \to 0$ (A) by Tauber's first theorem that $\tau_n^{\alpha+\theta+1} = o(1)$. This equal to $\tau_n^{\alpha+1} \to 0$ (C,θ) and then we get $\tau_n^{\alpha+1} \to 0$ (A). We get

$$(n\Delta)_{m-1} \tau_n^{\alpha+m} \to 0 \ (A) \tag{3.6}$$

by the identity

$$(\alpha+2)(\tau_n^{\alpha+1} - \tau_n^{\alpha+2}) = n\Delta\tau_n^{\alpha+2}. \tag{3.7}$$

Then from (3.1) and (3.6) by Tauber's first theorem, we have

$$(n\Delta)_{m-1}\tau_n^{\alpha+m} = o(1).$$

Continuing in this way, we have

$$n\Delta\tau_n^{\alpha+2} = o(1).$$

It follows from this equation and $\tau_n^{\alpha+2} \to 0\ (A)$ by Tauber's first theorem that

$$\tau_n^{\alpha+2} = o(1).$$

Then we get $\tau_n^{\alpha+1} = o(1)$ by identity (3.7), this give us $\tau_n^{\alpha+\theta+1} = o(1)$.

From $u_n^{\alpha+\theta} \to \xi\ (A)$ and $\tau_n^{\alpha+\theta+1} = n\Delta u_n^{\alpha+\theta+1} = o(1)$ by Tauber's first theorem, we get $u_n^{\alpha+\theta+1} \to \xi$ then we have $u_n^{\alpha} \to \xi\ (C, \theta+1)$ finally we obtain $u_n^{\alpha} \to \xi\ (A)$.

Therefore $u_n^{\alpha+1} \to \xi\ (A)$.

Similarly, from $\tau_n^{\alpha+1} = n\Delta u_n^{\alpha+1} = o(1)$ and from $u_n^{\alpha+1} \to \xi\ (A)$ by Tauber's first theorem, we get $u_n^{\alpha+1} \to \xi$.

Then we get

$$u_n^{\alpha} \to \xi$$

from the identity

$$(\alpha+1)(u_n^{\alpha} - u_n^{\alpha+1}) = \tau_n^{\alpha+1}.$$

□

Proof of Theorem 2. Let θ be a positive integer such that $\alpha + \theta \geq \beta$. By hypothesis, (u_n) is (A)(C, β) summable to ξ for any finite number ξ. Therefore, $s_n \to \xi\ (A)(C, \alpha+\theta)$ and $u_n^{\alpha+\theta+1} \to \xi\ (A)$. Then we have,

$$\begin{aligned} \tau_n^{\alpha+\theta+1} &\to 0\ (A), && (3.8)\\ n\Delta\tau_n^{\alpha+\theta+1} &\to 0\ (A),\\ &\vdots\\ (n\Delta)_m\,\tau_n^{\alpha+\theta+m} &\to 0\ (A). \end{aligned}$$
$$(3.9)$$

From the following equation

$$\sigma_n^1\left(\tau_n^{\alpha+\theta}\right) = \frac{1}{\alpha+\theta+2}\tau_n^{\alpha+\theta+1} + \left(1 - \frac{1}{\alpha+\theta+2}\sigma_n^{(1)}\left(\tau^{\alpha+\theta+1}\right)\right)$$

and (3.8) we get

$$(C,1)\left(\tau_n^{\alpha+\theta}\right) \to 0\ (A)$$

and similarly

$$\begin{aligned}(C,1)\left(n\Delta\tau_n^{\alpha+\theta+1}\right) &\rightarrow 0\ (A),\\ &\vdots\\ (C,1)\left((n\Delta)_m\tau_n^{\alpha+\theta+m}\right) &\rightarrow 0\ (A).\end{aligned}$$

If we do similar operations, we obtain

$$\begin{aligned}(C,\theta)(C,1)\left(\tau_n^{\alpha}\right) &\rightarrow 0\ (A)\\ (C,\theta)(C,1)\left(\tau_n^{\alpha+1}\right) &\rightarrow 0\ (A)\end{aligned}$$

and similarly

$$\begin{aligned}(C,\theta)(C,1)\left(n\Delta\tau_n^{\alpha+1}\right) &\rightarrow 0\ (A)\\ &\vdots\\ (C,\theta)(C,1)\left((n\Delta)_m\tau_n^{\alpha+m}\right) &\rightarrow 0\ (A).\end{aligned}$$

In (3.2), the Lemma 2 is applied with $p = 1$ and if (3.2) is written like the following inequality

$$\lim_{\lambda\rightarrow 1^+}\liminf_{n\rightarrow\infty}\min_{n<t\leq[\lambda n]}\frac{1}{n}\left(\sum_{k=0}^{m}(k\Delta)_m\tau_k^{\alpha+m}-\sum_{k=0}^{n}(k\Delta)_n\tau_k^{\alpha+n}\right)\geq 0$$

we get

$$\frac{1}{n+1}\sum_{k=0}^{m}(k\Delta)_m\tau_k^{\alpha+m}\geq -C$$

so we have $(C,1)(n\Delta)_m\tau_n^{\alpha+m}\geq -C$. Then we have

$$(C,\theta)(C,1)(n\Delta)_m\tau_n^{\alpha+m}\geq -C. \tag{3.10}$$

From (3.10), (3.10), we get $(C,\theta)(C,1)(n\Delta)_m\tau_n^{\alpha+m}\rightarrow 0(C,1)$ by Karamata's main theorem's result. This give us

$$(n\Delta)_m\tau_n^{\alpha+m}\rightarrow 0(C,\theta+2)$$

and then

$$(n\Delta)_m\tau_n^{\alpha+m}\rightarrow 0(A) \tag{3.11}$$

From Lemma 3 in Badiozzaman [14], (3.11) and (3.2), we have

$$(n\Delta)_m\tau_n^{\alpha+m}\rightarrow 0(C,1)$$

In that case we get

$$(C,1)(n\Delta)_m\tau_n^{\alpha+m}=(n\Delta)_m^{\alpha+m+1}=o(1).$$

Thus we get condition of theorem 1.

□

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